SOLUTION OF PROBLEM 226.

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PUTTING $\frac{2k\pi}{n} = \theta$, $x = \cos\theta$ in the first of the required equations, and $x = \sin\theta$ in the second. In either case, $\sin n\theta = 0$, and $\cos n\theta = 1$; hence if we can express either of these quantities in a finite number of terms involving powers of $\cos\theta$ we shall have the first of the required equations, and if we express either of them in terms of $\sin\theta$, we have the second equation. Now $\sin n\theta$ cannot be expressed in a finite number of terms involving $\cos\theta$; but for $\cos n\theta$ we have (See Chauvenet's Trig., eq. 460, page 138),

$$\cos n\theta = \cos\frac{n\pi}{2}\left[1 - \frac{n^2}{2}\cos^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\cos^4\theta + \ldots\right]$$
$$+\sin\frac{n\pi}{2}\left[n\cos\theta - \frac{n(n^2-1^2)}{1.2.3}\cos^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\cos^5\theta..\right].$$

Substituting $\cos\theta = x$ we have, when n is an odd integer,

$$0 = -1 + (-1)^{\frac{n-1}{2}}\left[nx - \frac{n(n^2-1^2)}{1.2.3}x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 - \ldots \right.$$
$$\left. + (-1)^{\frac{n-1}{2}}\frac{n(n^2-1^2)\ldots[n^2-(n-2)^2]}{n!}x^n\right], \quad (1)$$

but when n is even we have

$$0 = -1 + (-1)^{\frac{n}{2}}\left[1 - \frac{n^2}{2}x^2 + \frac{n^2(n^2-2^2)}{4!}x^4 - \ldots \right.$$
$$\left. + (-1)^{\frac{n}{2}}\frac{n^2(n^2-2^2)\ldots[n^2-(n-2)^2}{n!}x^n\right]. \quad (2)$$

Equations (1) and (2) are therefore the equations whose roots are

$$x = \cos\frac{2k\pi}{n}.$$

For the second equation $x = \sin\theta$, and we are required to express either $\cos n\theta$ or $\sin n\theta$ in terms of $\sin\theta$. We have

$$\cos n\theta = 1 - \frac{n^2}{2}\sin^2\theta + \frac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta - \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}\sin^6\theta + \ldots$$

$$\sin n\theta = n\sin\theta - \frac{n(n^2-1^2)}{1.2.3}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta - \ldots.$$

When n is odd, the expression for $\sin n\theta$ terminates, and we have by substitution

$$0=x-\frac{n^2-1^2}{1.2.3}x^3+\frac{(n^2-1^2)(n^2-3^2)}{5\,!}x^5-\ldots$$
$$+(-1)^{\frac{n-1}{2}}\frac{(n^2-1^2)(n^2-3^2)\ldots[n^2-(n-2)^2]}{n\,!}x^n, \quad (3)$$

but when n is even the expression for $\cos\theta$ terminates, and we have, on substitution,

$$0=x^2-\frac{n^2-2^2}{3.4}x^4+\frac{(n^2-2^2)(n^2-4^2)}{3.4.5.6}x^6-\ldots$$
$$-(-1)^{\frac{n}{2}}\frac{(n^2-2^2)\ldots[n^2-(n-2)^2]}{3.4.5\ldots.n}x^n. \quad (4)$$

Equations (3) and (4) are therefore those whose roots are

$$x=\sin\frac{2k\pi}{n}.$$

The several roots are in each case found by giving to k the values 0, 1, 2, ... $n-1$. Putting $k=0$ we see that $x=1$ is a root of equations (1) and (2), whence we have the numerical series, when n is odd,

$$(-1)^{\frac{n-1}{2}}=n-\frac{n(n^2-1^2)}{1.2.3}+\frac{n(n^2-1^2)(n^2-3^2)}{5\,!}-\ldots.$$

and when n is even,

$$(-1)^{\frac{n}{2}}=1-\frac{n^2}{2}+\frac{n^2(n^2-2^2)}{4\,!}-\frac{n^2(n^2-2^2)(n^2-4^2)}{6\,!}+\ldots.$$

Since $\cos\frac{2k\pi}{n}=\cos\frac{2(n-k)\pi}{n}$ the roots (exclusive of unity) of eq. (1) occur in equal pairs, and the equation must be of the form

$$(x-1)[F_1(x)]^2=0,$$

and for a like reason equation (2) must have the form

$$(x^2-1)[F_2(x)]^2=0.$$

The equations $F_1(x)=0$ and $F_2(x)=0$ are therefore those whose roots are the *different* values of x, exclusive of 1 and -1, which are represented by $\cos(2k\pi\div n)$.

I proceed to find the form of the functions $F_1(x)$ and $F_2(x)$. Observe in the first place that we may, in any one of the equations, change n to $-n$, in each of the coefficients; and also that, since in (1) and (2) the degree of each term in n is the same as its degree in x, the same will be true in the quotients when we divide these equations by $-1+x$ or $-1+x^2$, and also in the square roots of the results, that is in the functions F_1 and F_2.

Assuming n to have the form $4n'+1$ in eq. (1) we may give to n the successive values $1, -3, 5, -7, 9$ etc., and by changing the sign of n we shall

have the form which the function takes when n is of the form $4n' + 3$, which form may be denoted by F_3, the admissible values of n being -1, 3, -5, 7 etc. It is obvious that the first term of $F_1(x)$ is 1; and, since the expression must reduce to its first term when $n = 1$, every term after the first must contain the factor $n-1$. In like manner all terms after the second must contain the factor $n + 3$, since when $n = -3$ all terms after the second must vanish; and so on. Thus we may assume

$$F_1(x) = 1+A(n-1)x+B(n-1)(n+3)x^2+\ldots\ldots$$

in which A, B, etc., are numerical factors, since the degree of each term in n is to be the same as its degree in x. Then the expression $F_3(x)$, found by changing n to $-n$, is

$$F_3(x) = 1-A(n+1)x+B(n+1)(n-3)x^2-\ldots\ldots$$

In either case the last term contains $x^{\frac{n-1}{2}}$, and $\frac{n-1}{2}$ factors which are successive multiples of 4, the least being 4; hence, denoting the numerical factor by L, the value of this term is

$$L\,.\,4^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)!\,x^{\frac{n-1}{2}}.$$

Now this term is the square root of the last term of $[F_1(x)]^2$ which is the result of dividing the second member of (1) by $-1+x$. The latter term may be written in the form

$$\frac{(2n-2)(2n-4)\ldots\ldots(n+1)(n-1)\ldots\ldots2}{(n-1)\,!}\,x^{n-1},$$

$$= 2^{n-1}\,x^{n-1},$$

the square root of which is $\pm\,2^{\frac{n-1}{2}}\,x^{\frac{n-1}{2}}$. Theoefore

$$L\,.\,4^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)!\,x^{\frac{n-1}{2}} = \pm 2^{\frac{n-1}{2}}\,x^{\frac{n-1}{2}},$$

whence

$$L = \pm\frac{1}{2^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)!},$$

which gives the value of each coefficient, since any term may be made the last term in one or other of the expressions $F_1(x)$ and $F_3(x)$. To determine the signs of the terms, observe that the coefficient of the term preceding the last in $[F_1(x)]^2$ is the same as that of the last term, and is therefore positive; hence the last two terms of $F_1(x)$ or $F_3(x)$ have the same sign. Then since $F_1(x)$ contains an odd number of terms A and B must have the same sign, and so must C and D, E and F, etc., and since $F_3(x)$ contains an even number of terms, 1 and $-A$, B and $-C$, etc., must have the same

sign. Thus we have when $n = 4n'+1$

$$F_1(x) = 1-\frac{n-1}{2}x-\frac{(n-1)(n+3)}{2^2.2!}x^2+\frac{(n-1)(n+3)(n-5)}{2^3.3!}x^3+\ldots$$

and when $n = 4n'+3$

$$F_3(x) = 1+\frac{n+1}{2}x-\frac{(n+1)(n-3)}{2^2.2!}x^2-\frac{(n+1)(n-3)(n+5)}{2^3.3!}x^3+\ldots$$

Eq. (2) takes different forms when n is of the form $4n'+2$, and when it is of the form $4n'$. In the first case, all the pairs of equal roots occur in couples having numerically equal positive and negative values, so that the equation may be written in the form $(x^2-1)[F_2(x^2)]^2 = 0$; but in the second case there is a single pair of zero roots, so that the form of the equation is $x^2(x^2-1)[F_4(x^2)]^2 = 0$.

When $n = 4n'+2$, eq. (2) becomes

$$0 = -1+\frac{n^2}{2.2}x^2-\frac{n^2(n^2-2^2)}{2.4!}x^4+\ldots.+\frac{n^2(n^2-2^2)\ldots[n^2-(n-2)^2]}{2.n!}x^n$$

in which n admits of the values $\pm\, 2$, $\pm\, 6$, $\pm\, 10$, etc. The first term of $F_2(x^2)$ is unity; and, since the expression must reduce to its first term when $n = \pm 2$, all the terms beyond the first must contain n^2-2^2 as a factor. In like manner, all the terms beyond the second must contain $n^2 - 6^2$, etc. Thus we may assume

$$F_2(x^2) = 1+A(n^2-2^2)x^2+B(n^2-2^2)(n^2-6^2)x^4+\ldots+L(n^2-2^2)\ldots \times[n^2-(n-4)^2]x^{\frac{n-2}{2}},$$

in which A, B, etc., have numerical values. The last term may be written in the form

$$L.(2n-4)(2n-8)\ldots(n+2)(n-2)\ldots 4x^{\frac{n-2}{2}} = L.4^{\frac{n-2}{2}}\left(\frac{n-2}{2}\right)!x^{\frac{n-2}{2}}.$$

But the last term of the quotient, when the above form of eq. (2) is divided by $-1+x^2$, is

$$\frac{(2n-2)(2n-4)\ldots(n+2)n(n-2)\ldots 2}{2.(n-1)!}x^{n-2} = 2^{n-2}x^{n-2},$$

the square root of which is $\pm 2^{\frac{n-2}{2}}x^{\frac{n-2}{2}}$. Hence

$$L.4^{\frac{n-2}{2}}\left(\frac{n-2}{2}\right)!x^{\frac{n-2}{2}} = \pm 2^{\frac{n-2}{2}}x^{\frac{n-2}{2}},$$

and

$$L = \pm\frac{1}{2^{\frac{n-2}{2}}\left(\frac{n-2}{2}\right)!},$$

which gives the form of the numerical factors. It is easily shown that the last term but one of $F_2(x^2)$ is negative, hence the last two terms of $F_2(x^2)$ are always of opposite signs, and we have

$$F_2(x^2) = 1 - \frac{n^2-2^2}{2^2.2!}x^2 + \frac{(n^2-2^2)(n^2-6^2)}{2^4.4!}x^4 - \ldots$$

When n is of the form $4n'$, eq. (2) becomes

$$0 = -x^2 + \frac{n^2-2^2}{3.4}x^4 - \frac{(n^2-2^2)(n^2-4^2)}{3.4.5.6}x^6 + \ldots + \frac{2(n^2-2^2)\ldots[n^2-(n-2)^2]}{n!}x^n,$$

in which n admits of the values ± 4, ± 8, etc., the form of the equation being $x^2(x^2-1)[F_4(x^2)]^2 = 0$, so that the degree of each term of $F_4(x^2)$ in n will still be the same as the degree in x. The first term of $F_4(x^2)$ is unity, and it may be assumed in the form

$$F_4(x^2) = 1 + A(n^2-4^2)x^2 + B(n^2-4^2)(n^2-8^2)x^4 + \ldots + L(n^2-4^2)\ldots\ldots \times [n^2-(n-4)^2]x^{\frac{n-4}{2}},$$

A, B, etc., having numerical values. In this case, the last term is

$$L.(2n-4)(2n-8)\ldots(n+4)(n-4)\ldots 4.x^{\frac{n-4}{2}} = L\frac{4^{\frac{n-2}{2}}\left(\frac{n-2}{2}\right)!}{n}x^{\frac{n-4}{2}}.$$

But the last term of $[F_4(x)]^2$ may be written

$$\frac{2(2n-2)(2n-4)\ldots\ldots(n+2)n(n-2)\ldots\ldots 2}{n\,.\,n!}x^{n-4}$$

$$= \frac{2^n(n-1)!}{n^2(n-1)!}x^{n-4} = \frac{2^n}{n^2}x^{n-4},$$

the square root of which is $\pm\frac{2^{\frac{1}{2}n}}{n}x^{\frac{n-4}{2}}$. Hence

$$L\frac{4^{\frac{n-2}{2}}\left(\frac{n-2}{2}\right)!}{n}x^{\frac{n-4}{2}} = \pm\frac{2^{\frac{1}{2}n}}{n}x^{\frac{n-4}{2}}, \text{ and } L = \pm\frac{1}{2^{\frac{n-4}{2}}\left(\frac{n-2}{2}\right)!}.$$

The terms are of alternate signs, for the reason already given in the case of F_2, hence we have

$$F_4(x^2) = 1 - \frac{n^2-4^2}{2^2.3!}x^2 + \frac{(n^2-4^2)(n^2-8^2)}{2^4.5!}x^4 - \ldots.$$

Of equation (3) we can only infer, what is obvious on inspection, that it takes the form $xF(x^2) = 0$; but eq. (4) takes the form $x^2[F(x^2)]^2 = 0$ when n has the form $4n'+2$, and the form $x^2(x^2-1)[F(x^2)]^2 = 0$, when n has the form $4n'$. In the latter case, the equation becomes identical with eq. (2) in the like case, and in the former case, we find by the method pursued above

$$F(x^2) = 1 - \frac{n^2-2^2}{2^2.3!}x^2 + \frac{(n^2-2^2)(n^2-6^2)}{2^4.5!}x^4 - \ldots.$$